Double integral

Calculate the following double integral:

$$\int_0^3 \int_4^5 \frac{(4x+1)3y}{x^2-5x+6}\,dx\,dy$$

Solution

Ignore the integration limits and focus on the inner integral. First, we write $4x+1$ as $2(2x-5)+11$ and separate:

$$\int \frac{4x+1}{x^2-5x+6}\,dx = \int \left(\frac{2(2x-5)}{x^2-5x+6} + \frac{11}{x^2-5x+6}\right) dx$$

We apply linearity:

$$= 2\int \frac{2x-5}{x^2-5x+6}\,dx + 11\int \frac{1}{x^2-5x+6}\,dx$$

Now solving:

$$2\int \frac{2x-5}{x^2-5x+6}\,dx$$

Substituting $u = x^2-5x+6 \to du = (2x-5)dx$:

$$= 2\int \frac{1}{u}\,du$$

This is a standard integral:

$$= 2\ln(u)$$

Undo the substitution $u = x^2-5x+6$:

$$= 2\ln(|x^2-5x+6|)$$

Now solving:

$$11\int \frac{1}{x^2-5x+6}\,dx$$

Factor the denominator:

$$= 11\int \frac{1}{(x-3)(x-2)}\,dx$$

Perform partial fraction decomposition:

$$= 11\int \left(\frac{1}{x-3} - \frac{1}{x-2}\right) dx$$

Apply linearity:

$$= 11\int \frac{1}{x-3}\,dx - 11\int \frac{1}{x-2}\,dx$$

Now solving:

$$11\int \frac{1}{x-3}\,dx$$

Substituting $u = x-3 \to du = dx$:

$$= 11\int \frac{1}{u}\,du$$

Use the previous result:

$$= 11\ln(u)$$

Undo the substitution $u = x-3$:

$$= 11\ln(|x-3|)$$

Now solving:

$$11\int \frac{1}{x-2}\,dx$$

Substituting $u = x-2 \to du = dx$:

$$= 11\int \frac{1}{u}\,du$$

Use the previous result:

$$= 11\ln(u)$$

Undo the substitution $u = x - 2$:

$$= 11\ln(|x-2|)$$

Insert the solved integrals:

$$11\int \frac{1}{x-3}\,dx - 11\int \frac{1}{x-2}\,dx = 11\ln(|x-3|) - 11\ln(|x-2|)$$

Insert the solved integrals:

$$2\int \frac{2x-5}{x^2-5x+6}\,dx + 11\int \frac{1}{x^2-5x+6}\,dx = 2\ln(|x^2-5x+6|) - 11\ln(|x-2|) + 11\ln(|x-3|)$$

Using this result in the original problem:

$$\int_0^3 3y\left(2\ln(|x^2-5x+6|) - 11\ln(|x-2|) + 11\ln(|x-3|)\right)\Big|_4^5\,dy$$

$$\int_0^3 3y\int_4^5 \frac{(4x+1)}{x^2-5x+6}\,dx\,dy = \int_0^3 3y\,(-0.8766+6.238)\,dy = 5.3622\int_0^3 y\,dy$$

$$16.0852\int_0^3 y\,dy = 16.0852\left.\frac{y^2}{2}\right|_0^3 = 16.0852\left(\frac{3^2}{2}\right) = 72.38$$